ELEN E3401: Electromagnetics

Spring 2025

Prof. Keren Bergman

Lecture #20

COLUMBIA ENGINEERING
The Fu Foundation School of Engineering and Applied Science

Wave polarization

Polarization – describes <u>locus</u> traced by tip of $\vec{E}$ vector in space (within plane perpendicular to direction of propagation) as function of time

Most general → elliptical polarization

circular   linear

z – components of E, $H = 0$ if $\hat{z}$ is propagation direction

$\tilde{E}(z)$ with $\hat{z}$ propagation: $\quad \tilde{E}(z) = \hat{x}\tilde{E}_x(z) + \hat{y}\tilde{E}_y(z)$

$\tilde{E}_x = E_{x0}e^{-jkz} \qquad \tilde{E}_y = E_{y0}e^{-jkz}$ (Only propagate in +z direction)

$E_{x0}, E_{y0} \rightarrow$ in general composed with magnitude and phase

Phase is defined relative to reference, such as z=0, t=0.

Polarization depends on phase and amplitude of E_{x0} relative to E_{y0}

Wave polarization

We set phase of E_{x0} as zero for reference

δ = phase of E_{y0} relative to E_{x0} = phase difference $\begin{cases} E_{x0} = a_x & a_x = |E_{x0}| \geq 0 \\ E_{y0} = a_y e^{j\delta} & a_y = |E_{y0}| \geq 0 \end{cases}$

$$\tilde{E}(z) = \hat{x}\tilde{E}_x(z) + \hat{y}\tilde{E}_y(z) = (\hat{x}a_x + \hat{y}a_y e^{j\delta})e^{-jkz}$$

Instantaneous field: $\vec{E}(z,t) = Re\left[\tilde{E}(z)e^{j\omega t}\right]$

$$\vec{E}(z,t) = \hat{x}a_x\cos(\omega t - kz) + \hat{y}a_y\cos(\omega t - kz + \delta)$$

Magnitude of $\vec{E}(z,t)$:

$$\left|\vec{E}(z,t)\right| = \sqrt{E_x^2(z,t) + E_y^2(z,t)} = ({a_x}^2 cos^2(\omega t - kz) + {a_y}^2 cos^2(\omega t - kz + \delta))^{1/2}$$

The direction of $\vec{E}(z,t)$ at specific position, z is given by "inclination angle" ψ :

$$\psi(z,t) = tan^{-1}\left[\frac{E_y(z,t)}{E_x(z,t)}\right]$$

Linear Polarization

Linearly polarized if at fixed z the tip of $\vec{E}(z,t)$ traces a straight line as function of time

$E_x(z,t)$ and $E_y(z,t)$ are in phase ($\delta = 0$) or out of phase ($\delta = \pi$)

$$\vec{E}(0,t) = \left(\hat{x}a_x + \hat{y}\,a_y\right)\cos(\omega t - kz) \quad \text{in phase}$$

$$\vec{E}(0,t) = \left(\hat{x}a_x - \hat{y}\,a_y\right)\cos(\omega t - kz) \quad \text{out of phase}$$

Consider **out of phase** case:

$$\left|\vec{E}(z,t)\right| = \sqrt{{a_x}^2 + {a_y}^2}\cos(\omega t - kz)$$

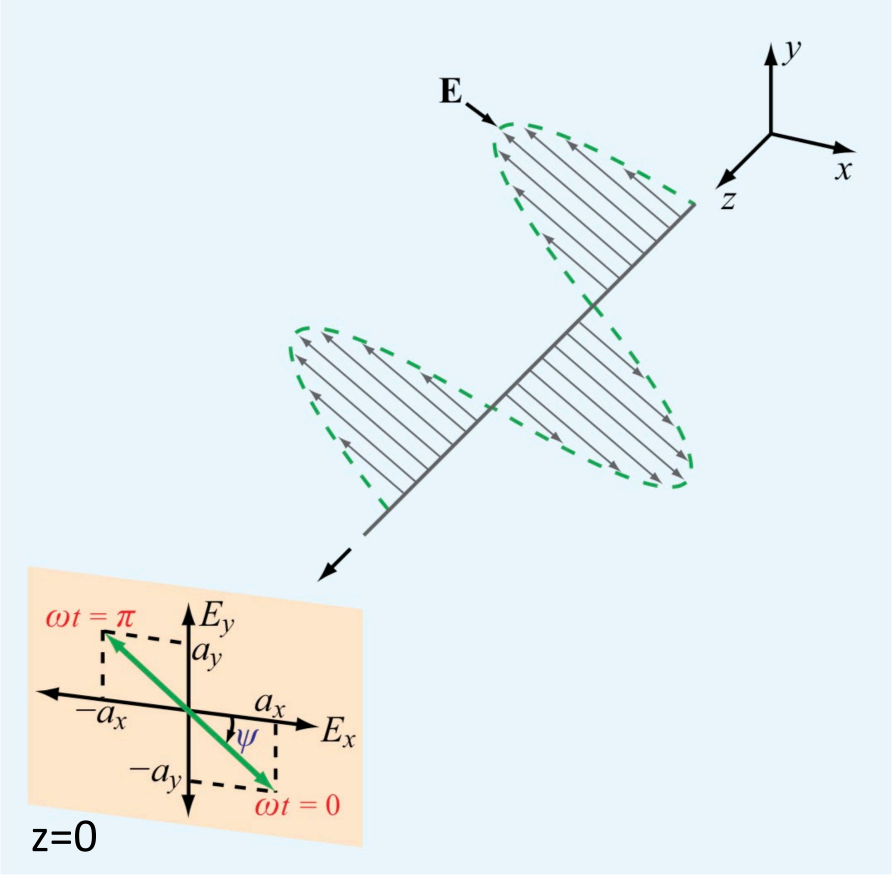
$$\psi = tan^{-1}\left[\frac{-a_y}{a_x}\right]$$

ψ is independent of both z and t

Linear Polarization

Linear polarization travelling $+z$

$|\vec{E}(z,t)| = \sqrt{{a_x}^2 + {a_y}^2}\cos(\omega t - kz) \quad \psi = tan^{-1}\left[\frac{-a_y}{a_x}\right]$



<u>Consider the electric field at $z = 0$:</u>

At $t = 0$

$$|\vec{E}(0,0)| = \sqrt{{a_x}^2 + {a_y}^2}$$

At $\omega t = \frac{\pi}{2}$, $t = \frac{\pi}{2}\frac{1}{\omega}$

$$\left|\vec{E}\left(0, t = \frac{\pi}{2\omega}\right)\right| = 0$$

Then vector reverses direction at $\omega t = \pi$

$|\vec{E}| = \sqrt{{a_x}^2 + {a_y}^2}$ in 2nd quadrant

If $a_y = 0$, then $\psi = 0° or\ 180°$ and wave is x-polarized

If $a_x = 0$, then $\psi = 90° or\ -90°$and wave is y-polarized

Circular Polarization

$$\left|\vec{E}(z,t)\right| = \sqrt{E_x^2(z,t) + E_y^2(z,t)} = ({a_x}^2 cos^2(\omega t - kz) + {a_y}^2 cos^2(\omega t - kz + \delta))^{\ 1/2}$$

$$\psi(z,t) = tan^{-1}\left[\frac{E_y(z,t)}{E_x(z,t)}\right]$$

Consider $a_x = a_y$ and $\delta = \pm\frac{\pi}{2}$

Will have <u>left-hand circular</u> $\delta = \frac{\pi}{2}$, <u>right-hand circular</u> $\delta = -\frac{\pi}{2}$

<u>LHC:</u> $a_x = a_y = a$ and $\delta = \frac{\pi}{2}$

$$\tilde{E}(z) = \left(\hat{x}a + \hat{y}ae^{j\frac{\pi}{2}}\right)e^{-jkz} = a(\hat{x} + j\hat{y})e^{-jkz}$$

$$\vec{E}(z,t) = Re\left[\tilde{E}(z)e^{j\omega t}\right] = \hat{x}a\cos(\omega t - kz) + \hat{y}a\cos(\omega t - kz + \frac{\pi}{2})$$

$$\vec{E}(z,t) = Re\left[\tilde{E}(z)e^{j\omega t}\right] = \hat{x}a\cos(\omega t - kz) - \hat{y}a\sin(\omega t - kz)$$

Circular Polarization

LHC: $a_x = a_y = a$ and $\delta = \frac{\pi}{2}$

$$\tilde{E}(z) = \left(\hat{x}a + \hat{y}ae^{j\frac{\pi}{2}}\right)e^{-jkz} = a(\hat{x} + j\hat{y})e^{-jkz}$$

$$\vec{E}(z,t) = Re\left[\tilde{E}(z)e^{j\omega t}\right] = \hat{x}a\cos(\omega t - kz) + \hat{y}a\cos(\omega t - kz + \frac{\pi}{2})$$

$$\vec{E}(z,t) = Re\left[\tilde{E}(z)e^{j\omega t}\right] = \hat{x}a\cos(\omega t - kz) - \hat{y}a\sin(\omega t - kz)$$

$$\left|\vec{E}(z,t)\right| = \sqrt{E_x^2(z,t) + E_y^2(z,t)} = [a^2cos^2(\omega t - kz) + a^2sin^2(\omega t - kz)]^{1/2} = a$$

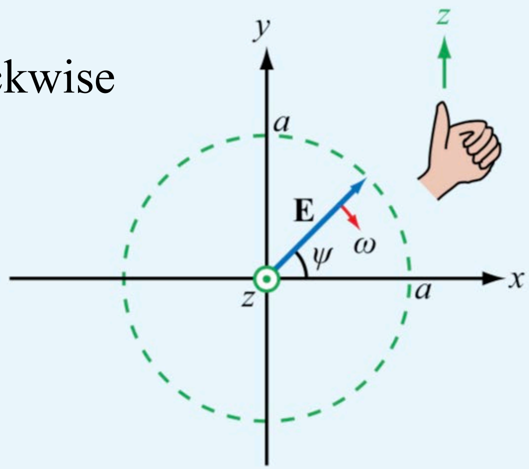
$$\psi(z,t) = tan^{-1}\left[\frac{E_y(z,t)}{E_x(z,t)}\right] = tan^{-1}\left[\frac{-a\sin(\omega t - kz)}{a\cos(\omega t - kz)}\right] = -(\omega t - kz)$$

Now $\left|\vec{E}\right|$ is independent of z, t and ψ depends on both z and t

At $z = 0$, $\psi = -\omega t \rightarrow$ inclination angle decreases with time

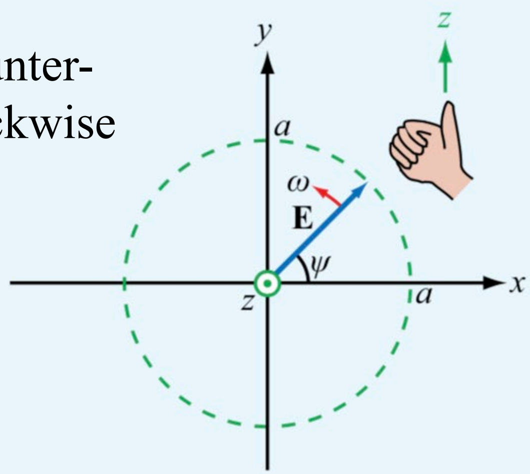
Circular Polarization

clockwise



(a) LHC polarization

Counter-clockwise



(b) RHC polarization

Thumb along $+\hat{z}$ direction of propagation – LH or RH then determines circular polarization

<u>RHC</u> $a_x = a_y = a$ and $\delta = -\frac{\pi}{2}$

$$\left|\vec{E}(z,t)\right| = a$$

$$\psi(z,t) = (\omega t - kz)$$

At z = 0, $\psi = \omega t$

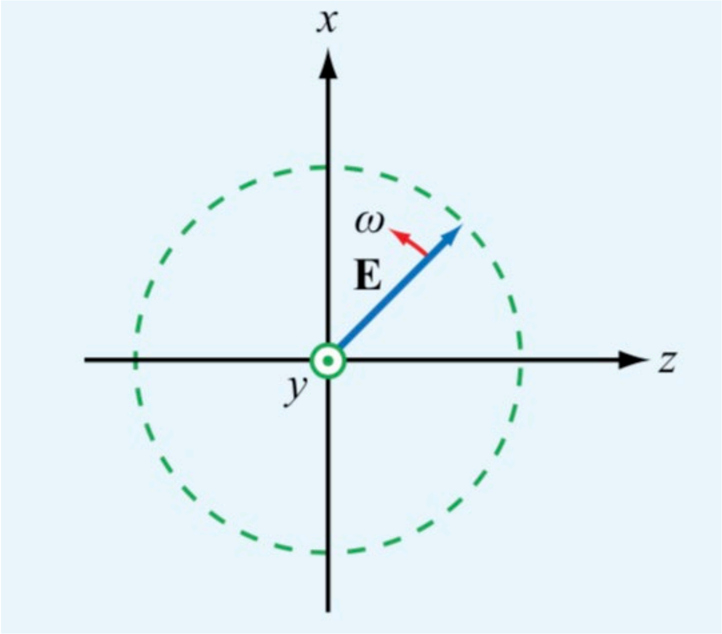
Example: RHC polarized wave

RHC – polarized plane wave with electric field magnitude = 3mV/m traveling in $+y$ direction in dielectric $\epsilon = 4\epsilon_0, \mu = \mu_0, \sigma = 0$

f = 100 MHz → **<u>obtain</u>** $\vec{E}(y,t)$ and $\vec{H}(y,t)$

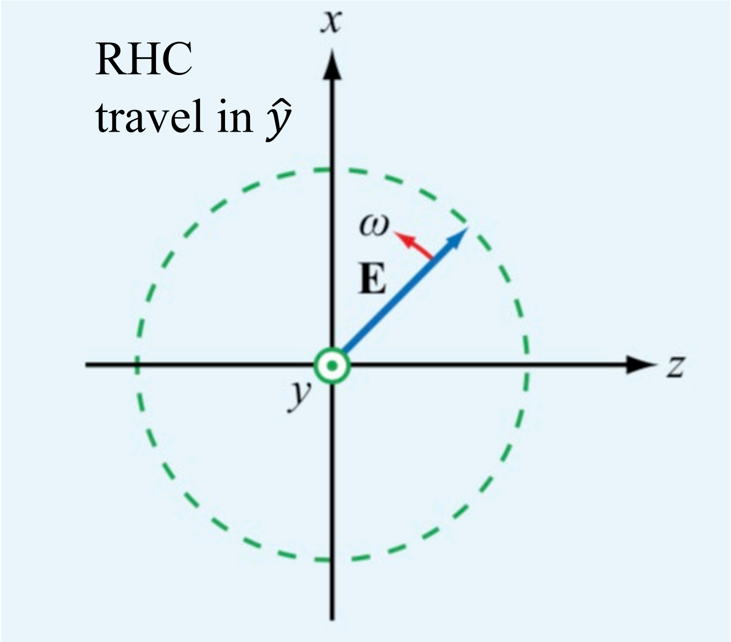
Since wave travels in $+y$ → $\hat{k} = \hat{y}$; plane wave must be in x-z plane

RHC travel in $\hat{y}$



Example: RHC polarized wave

$$\vec{E}(y,t) = \hat{z}a_z\cos(\omega t - ky) + \hat{x}a_x\cos(\omega t - ky + \delta)$$



$\tilde{E}(y)$: z-component phase = 0

x-component $\delta = -\frac{\pi}{2}$ phase shift

$a = 3$ (mv/m)

$$\tilde{E}(y) = \hat{z}\tilde{E}_z + \hat{x}\tilde{E}_x = \hat{z}ae^{-jky} + \hat{x}ae^{-j\frac{\pi}{2}}e^{-jky}$$

$$\tilde{E}(y) = (\hat{z} - \hat{x}j)3e^{-jky}$$

$$\tilde{H}(y) = \frac{1}{\eta}\hat{y} \times \tilde{E}(y) = \frac{1}{\eta}\hat{y} \times (\hat{z} - \hat{x}j)3e^{-jky}$$

$$\tilde{H}(y) = \frac{3}{\eta}(\hat{x} + \hat{z}j)e^{-jky}$$

Example: RHC polarized wave

$$\omega = 2\pi f = 2\pi \times 10^8 \ [rad/s]$$

$$k = \omega\sqrt{\mu\epsilon} = \omega\sqrt{\mu_0 4\epsilon_0} = 2\omega\sqrt{\mu_0\epsilon_0} \qquad k = \frac{2\omega}{c} = \frac{2\pi \times 10^8}{3 \times 10^8} = \frac{4}{3}\pi$$

$$\eta = \frac{\eta_0}{\sqrt{4}} \qquad \eta = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu_0}{\epsilon_0}}\frac{1}{\sqrt{4}} = \frac{\eta_0}{2} \sim \frac{120\pi}{2} = 60\pi$$

$$\vec{E}(y,t) = Re\left[\tilde{E}(y)e^{j\omega t}\right] = Re\left[(-\hat{x}j + \hat{z})3e^{-jky}e^{j\omega t}\right]$$

$$\boxed{\vec{E}(y,t) = 3[\hat{x}\sin(\omega t - ky) + \hat{z}\cos(\omega t - ky)] \ [mV/m]}$$

$$\vec{H}(y,t) = Re\left[\tilde{H}(y)e^{j\omega t}\right] = Re\left[\frac{3}{\eta}(\hat{z}j + \hat{x})e^{-jky}e^{j\omega t}\right]$$

$$\boxed{\vec{H}(y,t) = \frac{1}{20\pi}[\hat{x}\cos(\omega t - ky) - \hat{z}\sin(\omega t - ky)] \ [mA/m]}$$

Plane-wave propagation in lossy media

We start with wave equation

Recall our derivation, from time-dependent fields

$$\vec{\nabla} \times (\vec{\nabla} \times \tilde{E}) = -j\omega\mu\underbrace{(\vec{\nabla} \times \tilde{H})}$$

$$\underbrace{\vec{\nabla}\left(\vec{\nabla} \cdot \tilde{E}\right)}_{=0} - \nabla^2 \tilde{E}$$

$$\vec{\nabla} \times \tilde{H} = j\omega\epsilon_c \tilde{E} \qquad \epsilon_c = \epsilon - j\frac{\sigma}{\omega}$$

$$-\nabla^2 \tilde{E} = -j\omega\mu(j\omega\epsilon\tilde{E})$$

ϵ_c → complex

Homogeneous wave equation: $\nabla^2 \tilde{E} + \omega^2\mu\epsilon\tilde{E} = 0$

Define propagation constant: γ $\qquad \gamma^2 = -\omega^2\mu\epsilon_c$

$$\boxed{\nabla^2 \tilde{E} - \gamma^2 \tilde{E} = 0}$$

Lossless → $\epsilon_c = \epsilon$

$$\gamma^2 = -\omega^2\mu(\epsilon' - j\epsilon'') \qquad \epsilon' = \epsilon \qquad \epsilon'' = \frac{\sigma}{\omega}$$

$\gamma = \alpha + j\beta$ $\qquad \alpha \rightarrow$ attenuation constant $\qquad \beta \rightarrow$ phase constant

Plane-wave propagation in lossy media

$$(\alpha + j\beta\,)^2 = (\alpha^2 - \beta^2) + j2\alpha\beta = -\omega^2\mu\epsilon' + j\omega^2\mu\epsilon''$$

Real and imaginary parts must equal each other respectively:

$$(\alpha^2 - \beta^2) = -\omega^2\mu\epsilon'$$

$$2\alpha\beta = \omega^2\mu\epsilon''$$

Solve for α, β:

$$\alpha = \omega\left\{\frac{\mu\epsilon'}{2}\left[\sqrt{1+\left(\frac{\epsilon''}{\epsilon'}\right)^2} - 1\right]\right\}^{1/2}$$

Attenuation

[Np/m]

$$\beta = \omega\left\{\frac{\mu\epsilon'}{2}\left[\sqrt{1+\left(\frac{\epsilon''}{\epsilon'}\right)^2} + 1\right]\right\}^{1/2}$$

[rad/m]

phase

Plane-wave propagation in lossy media

Consider uniform plane wave, $\tilde{E} = \hat{x}\tilde{E}_x(z)$ traveling along z-direction

Wave equation becomes: $$\frac{d^2\tilde{E}_x(z)}{dz^2} - \gamma^2\tilde{E}_x(z) = 0$$

General solution will have $+z$ and $-z$ components

Assume only $+z$ solution: $$\tilde{E}(z) = \hat{x}\tilde{E}_x(z) = \hat{x}E_{x0}e^{-\gamma z}$$

$$\tilde{E}(z) = \hat{x}E_{x0}e^{-\alpha z}e^{-j\beta z}$$

Obtain associated magnetic field, $\widetilde{H}$: $\vec{\nabla} \times \tilde{E} = -j\omega\mu\widetilde{H}$ or $\widetilde{H} = \dfrac{\hat{k} \times \tilde{E}}{\eta_c}$

Intrinsic impedance, complex for lossy medium

Plane-wave propagation in lossy media

$$\tilde{E}(z) = \hat{x}E_{x0}e^{-\alpha z}e^{-j\beta z} \qquad \tilde{H}(z) = \hat{y}\tilde{H}_y(z) = \frac{\hat{y}\tilde{E}_x(z)}{\eta_c} = \hat{y}\frac{E_{x0}}{\eta_c}e^{-\alpha z}e^{-j\beta z}$$

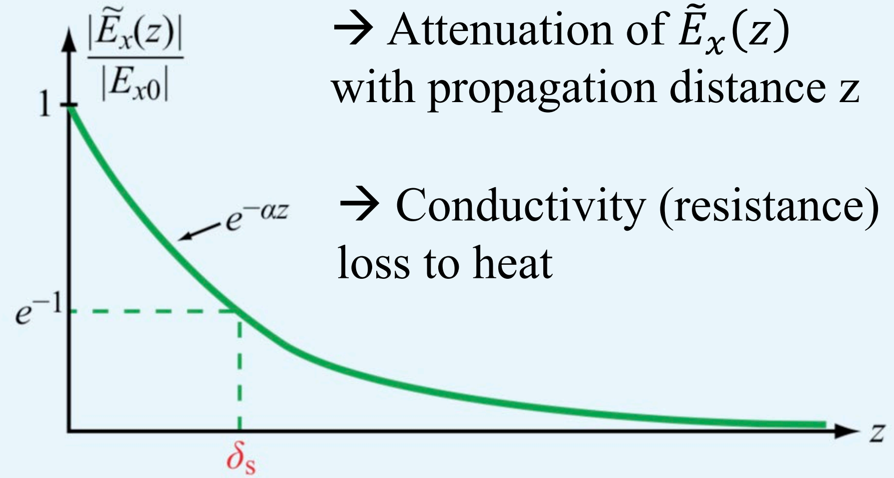
$$\eta_c = \sqrt{\frac{\mu}{\epsilon_c}} = \sqrt{\frac{\mu}{\epsilon'}}\left(1 - j\frac{\epsilon''}{\epsilon'}\right)^{-1/2}$$

For <u>lossy</u> medium, $\vec{E}(z,t)$ <u>no longer</u> in phase with $\vec{H}(z,t)$ with η_c complex (will see in examples)

Examine magnitude of $\tilde{E}_x(z)$:

$$|\tilde{E}_x(z)| = \left|E_{x0}e^{-\alpha z}e^{-j\beta z}\right|$$

$$|\tilde{E}_x(z)| = |E_{x0}|e^{-\alpha z}$$



→ Attenuation of $\tilde{E}_x(z)$ with propagation distance z

→ Conductivity (resistance) loss to heat

Plane-wave propagation in lossy media

$$\tilde{H}_y = \frac{\tilde{E}_x}{\eta_c} \longrightarrow \quad |\tilde{H}_y| \text{ will also decrease exponentially, } e^{-\alpha z}$$

Energy carried by EM wave is dissipated to heat due to conductivity of medium

Define distance, $z = \delta_s$ such that $\delta_s = \frac{1}{\alpha}$

Wave amplitude decreases by $1/e \approx 0.37$ at $z = \delta_s$

For $z = 3\delta_s$, field amp < 5%

For $z = 5\delta_s$, field amp < 1%

δ_s = skin depth → describes how deeply EM wave can penetrate conducting medium

Plane-wave propagation in lossy media

Perfect dielectric:

$$\sigma = 0, \epsilon'' = 0 \rightarrow \alpha = 0 \text{ and } \delta_s = \infty$$

Free space, plane wave can propagate indefinitely with no amplitude losses

Perfect conductor:

$$\sigma = \infty \text{ and } \delta_s = 0$$

EM wave cannot propagate through outer conductor of coax cable, prevents energy from leaking out. (shields)

Plane-wave propagation in lossy media

Apply to general – linear, isotropic, homogeneous media

$$\gamma = \alpha + j\beta \qquad \epsilon' = \epsilon \qquad \epsilon'' = \frac{\sigma}{\omega}$$

$$\alpha = \omega\left\{\frac{\mu\epsilon'}{2}\left[\sqrt{1+\left(\frac{\epsilon''}{\epsilon'}\right)^2} - 1\right]\right\}^{1/2}$$

$$\beta = \omega\left\{\frac{\mu\epsilon'}{2}\left[\sqrt{1+\left(\frac{\epsilon''}{\epsilon'}\right)^2} + 1\right]\right\}^{1/2}$$

$$\eta_c = \sqrt{\frac{\mu}{\epsilon_c}} = \sqrt{\frac{\mu}{\epsilon'}}\left(1 - j\frac{\epsilon''}{\epsilon'}\right)^{-1/2}$$

Plane-wave propagation in lossy media

1. Perfect dielectric: ($\sigma = 0$) → reduce to lossless case

$$\alpha = 0, \beta = k = \omega\sqrt{\mu\epsilon} \qquad \eta_c = \eta$$

2. Lossy medium: $$\frac{\epsilon''}{\epsilon'} = \frac{\sigma}{\omega\epsilon}$$

(a) <u>Low Loss:</u> $$\frac{\epsilon''}{\epsilon'} \ll 1 \qquad \frac{\epsilon''}{\epsilon'} < \frac{1}{100}$$

(b) <u>Good conductor:</u> $$\frac{\epsilon''}{\epsilon'} \gg 1 \qquad \frac{\epsilon''}{\epsilon'} > 100$$

(c) <u>Quasi conductor</u>: $$\frac{1}{100} \leq \frac{\epsilon''}{\epsilon'} \leq 100$$

Low-loss dielectric

$$\gamma^2 = -\omega^2\mu(\epsilon' - j\epsilon'') \Rightarrow \gamma = j\omega\sqrt{\mu\epsilon'}\left(1 - j\frac{\epsilon''}{\epsilon'}\right)^{\frac{1}{2}}$$

Using binomial series we can approximate $(1-x)^{1/2} \approx 1 - \dfrac{x}{2}$ (for $|x| \ll 1$)

Low loss dielectric $\left|j\dfrac{\epsilon''}{\epsilon'}\right| \ll 1$:

$$\gamma \approx j\omega\sqrt{\mu\epsilon'}\left(1 - j\frac{\epsilon''}{2\epsilon'}\right)$$

$$\alpha \approx \frac{\omega\epsilon''}{2}\sqrt{\frac{\mu}{\epsilon'}} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}} \quad \text{[Np/m]}$$

$$\beta \approx \omega\sqrt{\mu\epsilon'} = \omega\sqrt{\mu\epsilon} \quad \text{[rad/m]}$$

(same as k for lossless)

Low-loss dielectric

$$\eta_c = \sqrt{\frac{\mu}{\epsilon_c}} = \sqrt{\frac{\mu}{\epsilon'}}\left(1 - j\frac{\epsilon''}{\epsilon'}\right)^{-1/2}$$

again apply binomial expansion: $(1 - x)^{-1/2} \approx 1 + \frac{x}{2}$

$$\eta_c \approx \sqrt{\frac{\mu}{\epsilon'}}\left(1 + j\frac{\epsilon''}{2\epsilon'}\right) = \sqrt{\frac{\mu}{\epsilon}}\left(1 + j\frac{\sigma}{2\omega\epsilon}\right)$$

Since $\frac{\epsilon''}{\epsilon'} = \frac{\sigma}{\omega\epsilon} < \frac{1}{100}$ → can ignore

$$\eta_c \approx \sqrt{\frac{\mu}{\epsilon}}$$

Good Conductor

$$\alpha = \omega\left\{\frac{\mu\epsilon'}{2}\left[\sqrt{1+\left(\frac{\epsilon''}{\epsilon'}\right)^2}-1\right]\right\}^{1/2} \qquad \beta = \omega\left\{\frac{\mu\epsilon'}{2}\left[\sqrt{1+\left(\frac{\epsilon''}{\epsilon'}\right)^2}+1\right]\right\}^{1/2}$$

For $\frac{\epsilon''}{\epsilon'} > 100$ can approximate:

$$\alpha \approx \omega\sqrt{\frac{\mu\epsilon''}{2}} = \omega\sqrt{\frac{\mu\sigma}{2\omega}} = \sqrt{\pi f\mu\sigma}\ [Np/m]$$

$$\beta = \alpha \approx \sqrt{\pi f\mu\sigma}\ [rad/m]$$

Good Conductor

$$\eta_c = \sqrt{\frac{\mu}{\epsilon'}}\left(1 - j\frac{\epsilon''}{\epsilon'}\right)^{-1/2}$$

$$\frac{\epsilon''}{\epsilon'} > 100$$

recall: $\sqrt{j} = \frac{1+j}{\sqrt{2}}$

$$\eta_c \approx \sqrt{\frac{j\mu}{\epsilon''}} = (1+j)\sqrt{\frac{\pi f \mu}{\sigma}} = (1+j)\frac{\alpha}{\sigma}$$

For perfect conductor → $\sigma = \infty$, $\alpha = \beta = \infty$ and $\eta_c = 0$

equivalent to short circuit in transmission line

Summary of propagation in materials

	Any Medium	Lossless Medium ($\sigma = 0$)	Low-loss Medium ($\varepsilon''/\varepsilon' \ll 1$)	Good Conductor ($\varepsilon''/\varepsilon' \gg 1$)	Units
$\alpha =$	$\omega \left[\frac{\mu\varepsilon'}{2} \left[\sqrt{1 + \left(\frac{\varepsilon''}{\varepsilon'}\right)^2} - 1 \right] \right]^{1/2}$	0	$\frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$	$\sqrt{\pi f \mu \sigma}$	(Np/m)
$\beta =$	$\omega \left[\frac{\mu\varepsilon'}{2} \left[\sqrt{1 + \left(\frac{\varepsilon''}{\varepsilon'}\right)^2} + 1 \right] \right]^{1/2}$	$\omega\sqrt{\mu\varepsilon}$	$\omega\sqrt{\mu\varepsilon}$	$\sqrt{\pi f \mu \sigma}$	(rad/m)
$\eta_c =$	$\sqrt{\frac{\mu}{\varepsilon'}} \left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)^{-1/2}$	$\sqrt{\frac{\mu}{\varepsilon}}$	$\sqrt{\frac{\mu}{\varepsilon}}$	$(1+j)\frac{\alpha}{\sigma}$	(Ω)
$u_p =$	ω/β	$1/\sqrt{\mu\varepsilon}$	$1/\sqrt{\mu\varepsilon}$	$\sqrt{4\pi f/\mu\sigma}$	(m/s)
$\lambda =$	$2\pi/\beta = u_p/f$	u_p/f	u_p/f	u_p/f	(m)

Notes: $\varepsilon' = \varepsilon$; $\varepsilon'' = \sigma/\omega$; in free space, $\varepsilon = \varepsilon_0$, $\mu = \mu_0$; in practice, a material is considered a low-loss medium if $\varepsilon''/\varepsilon' = \sigma/\omega\varepsilon < 0.01$ and a good conducting medium if $\varepsilon''/\varepsilon' > 100$.

Example: copper

Copper has:

$$\mu = \mu_0 = 4\pi \times 10^{-7} \left[\frac{H}{m}\right], \epsilon = \epsilon_0 = \frac{1}{36\pi} \times 10^{-9} \left[\frac{F}{m}\right], \sigma = 5.8 \times 10^{7} \left[\frac{S}{m}\right]$$

Assuming parameters do not change with frequency, over what spectral range is copper a good conductor?

Good conductor: $$\frac{\epsilon''}{\epsilon'} = \frac{\sigma}{\omega\epsilon} > 100$$

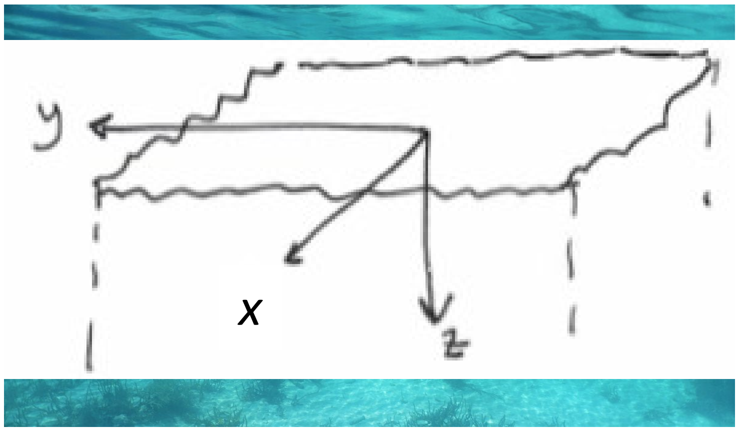
$$\omega = 2\pi \mathrm{f} < \frac{\sigma}{100\epsilon}$$

$$\mathrm{f} < \frac{\sigma}{200\pi\epsilon} = \frac{5.8 \times 10^{7}}{(200\pi)\left(\frac{1}{36\pi} \times 10^{-9}\right)} = 1.04 \times 10^{16}\ Hz$$

As long as f < ~10^{16} Hz, copper good conduction (UV light)

Example: plane wave in seawater

Consider a uniform plane wave in seawater



Plane wave in *x*-*y*

Propagate in +*z*

$$\epsilon_r = 80, \mu_r = 1, \sigma = 4\ S/m$$

At $z = 0$, $\vec{H}(0,t) = \hat{y}100\cos(2\pi 10^3 t + 15°)\ \ [mA/m]$

a) Obtain $\vec{E}(z,t)$ and $\vec{H}(z,t)$

b) Obtain depth where magnitude of $\vec{E}$ is 1% of z=0

Example: plane wave in seawater

a) Obtain $\vec{E}(z,t)$ and $\vec{H}(z,t)$

Since $\vec{H}$ is along $\hat{y}$ and propagates along $\hat{z}$ → $\vec{E}$ must be along $\hat{x}$:

General expressions for phasors: $\tilde{E}(z) = \hat{x}E_{x0}e^{-\alpha z}e^{-j\beta z}$

$$\tilde{H}(z) = \frac{\hat{y}E_{x0}}{\eta_c}e^{-\alpha z}e^{-j\beta z}$$

Now we determine α, β and η_c for seawater:

Evaluate $\frac{\epsilon''}{\epsilon'}$ We have $\omega = 2\pi f = 2\pi \times 10^3 \left[\frac{rad}{s}\right]$ $f = 1$ kHz

$$\frac{\epsilon''}{\epsilon'} = \frac{\sigma}{\omega\epsilon} = \frac{\sigma}{\omega\epsilon_r\epsilon_0} = \frac{4}{(2\pi \times 10^3)(80)\left(\frac{1}{36\pi} \times 10^{-9}\right)} = 9 \times 10^5 \gg 100$$

Seawater is a <u>Good</u>
conductor at 1 kHz

Example: plane wave in seawater

$$\alpha = \sqrt{\pi f \mu \sigma} = \sqrt{\pi \times 10^3 \times 4\pi \times 10^{-7} \times 4} = 0.126\ [Np/m]$$

$$\beta = \alpha = 0.126\ [rad/m]$$

$$\eta_c = (1+j)\frac{\alpha}{\sigma} = \left(\sqrt{2}e^{\frac{j\pi}{4}}\right)\frac{0.126}{4} = 0.044e^{\frac{j\pi}{4}}$$

$$E_{x0} = |E_{x0}|e^{j\varphi_0}$$

$$\vec{E}(z,t) = Re\left[\hat{x}|E_{x0}|e^{j\varphi_0}e^{-\alpha z}e^{-j\beta z}e^{j\omega t}\right]$$

$$\vec{E}(z,t) = \hat{x}|E_{x0}|e^{-0.126z}\cos(2\pi 10^3 t - 0.126z + \varphi_0)\ \ [V/m]$$

$$\vec{H}(z,t) = Re\left[\frac{\hat{y}|E_{x0}|e^{j\varphi_0}}{0.044e^{\frac{j\pi}{4}}}e^{-\alpha z}e^{-j\beta z}e^{j\omega t}\right]$$

$$= \hat{y}22.5|E_{x0}|e^{-0.126z}\cos(2\pi 10^3 t - 0.126z + \varphi_0 - 45°)\ \ \left[\frac{A}{m}\right]$$

Example: plane wave in seawater

At $z = 0$: $\vec{H}(0,t) = \hat{y}22.5|E_{x0}|\cos(2\pi 10^3 t + \varphi_0 - 45°)$ $\left[\frac{A}{m}\right]$

Compare with given:

$$\vec{H}(0,t) = \hat{y}100\cos(2\pi 10^3 t + 15°)\ \ [mA/m]$$

$$22.5|E_{x0}| = 100 \times 10^{-3} \rightarrow |E_{x0}| = 4.44\ mV/m$$

$$\varphi_0 - 45° = 15° \rightarrow \varphi_0 = 60°$$

$$\vec{E}(z,t) = \hat{x}4.44e^{-0.126z}\cos(2\pi 10^3 t - 0.126z + 60°)\ \ [mV/m]$$

$$\vec{H}(z,t) = \hat{y}100e^{-0.126z}\cos(2\pi 10^3 t - 0.126z + 15°)\ \ [mA/m]$$

Example: plane wave in seawater

b) Depth where magnitude of $\vec{E}$ is 1% of z=0

$$0.01 = e^{-0.126z}$$

$$z = \frac{\ln(0.01)}{-0.126} = 36.55\ m$$